

Hamiltonian structure of the fluid electron temperature gradient driven mode

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(Received 24 June 2003; accepted 16 October 2003)

Electron temperature gradient (ETG) driven turbulence can be simply described via reduced fluid equations. In this study, we describe the Hamiltonian structure of the simple curvature-driven ETG model, which is similar to the thermal Rossby wave model studied in atmospheric turbulence. Together with the Hamiltonian that is conserved, another constant of the motion has been identified. © 2004 American Institute of Physics. [DOI: 10.1063/1.1632497]

The electron temperature gradient (ETG) driven mode^{1,2} has been introduced as a quasi-fluid electron mode with dynamics akin to ITG³ (i.e., ion temperature gradient driven mode, but with $k_{\perp}\rho_i \gg 1$, $\tilde{n} = n_0 + e\Phi/T_i$). Recently, it has attracted attention because of its potential to explain the high levels of electron thermal transport that are being observed in transport barrier plasmas. The simplest model of toroidal ETG consists of a Hasegawa–Mima style vorticity equation coupled to the advection equation for the electron temperature, with constant background gradients:

$$(\partial_t + \hat{\mathbf{z}} \times \nabla \Phi \cdot \nabla)(1 - \nabla^2)\Phi + \partial_y(\Phi + P) + \nu \nabla^4 \Phi = 0, \quad (1)$$

$$(\partial_t + \hat{\mathbf{z}} \times \nabla \Phi \cdot \nabla)P - \chi \nabla^2 P - r \partial_y \Phi = 0. \quad (2)$$

In these equations, dimensionless drift wave variables are used such that

$$\Phi \rightarrow \frac{e\phi}{T_i \epsilon_{*i}}, \quad P \rightarrow \frac{\epsilon_B P_{e1}}{\epsilon_{*i}^2 P_{i0}},$$

$$t \rightarrow \Omega_e \epsilon_{*i} t, \quad \mathbf{x} \rightarrow \mathbf{x}' / \sqrt{\tau \rho_e},$$

$$\epsilon_{*i} = \frac{\sqrt{\tau \rho_e}}{L_n} = \frac{v_{*i}}{c_e}, \quad \epsilon_{*e} = -\frac{\sqrt{\tau \rho_e}}{L_p} = \frac{v_{*e}}{c_e},$$

$$\epsilon_B = \frac{\sqrt{\tau \rho_e}}{L_B} = \frac{v_B}{c_e}, \quad r = \frac{\epsilon_B \epsilon_{*e}}{\epsilon_{*i}^2},$$

where $\tau = T_i/T_e$. Notice also that by replacing $\epsilon_{*i} \rightarrow \epsilon_{*i}(1 - \epsilon_B/\epsilon_{*i})$ in the scalings above, one may obtain a model, which takes into account the effect of curvature drift on the scalar potential as well. This does not affect the form of the resulting equations [i.e., (1) and (2)].

Here we do not discuss the derivation of these equations. It is apparent, however, that this is a simplified model, with constant local background gradients and no magnetic fluctuations. Toroidal effects are modeled by a constant curvature drift v_B , and v_{*i} and v_{*e} are ion and electron diamagnetic drifts. The ion response is assumed to be completely adiabatic. For the inviscid limit of the model, we can write the equations in the form:

$$\begin{aligned} \partial_t \Phi &= (1 - \nabla^2)^{-1} \{ \Phi, \nabla^2 \Phi + x \} \\ &+ (1 - \nabla^2)^{-1} \left\{ \frac{P}{\sqrt{r}}, \sqrt{r} x \right\}, \end{aligned} \quad (3)$$

$$\partial_t \frac{P}{\sqrt{r}} = \left\{ \frac{P}{\sqrt{r}}, \Phi \right\} + \{ \sqrt{r} x, \Phi \}, \quad (4)$$

where $\{a, b\} = \hat{\mathbf{z}} \times \nabla a \cdot \nabla b$ is the Poisson bracket, and $(1 - \nabla^2)^{-1}$ is defined as $(1 - \nabla^2)^{-1}(1 - \nabla^2)a(\mathbf{x}) = a(\mathbf{x})$. For sinusoidal basis functions that are relevant for periodic domains, this operator can be represented by the kernel,

$$K(\mathbf{x} - \mathbf{x}') = \frac{1}{4\pi^2} \int \frac{e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')}}{1 + k^2} d^2\mathbf{k}.$$

The ETG model has a symplectic Hamiltonian structure similar to that of the Hasegawa–Mima (HM) equation.⁴

To show this, we start by defining the basis kets and bras as

$$|a\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \langle a| = (a_1, a_2)$$

and proceed with the inner product on the Hilbert space P :

$$\langle a|b\rangle = \sum_{i,j} \int a_i g_{ij}(b_j) d^2\mathbf{x}, \quad (5)$$

where g_{ij} plays the role of a metric operator, in analogy with the metric tensor, which defines the inner product in a similar manner. For this particular problem the choice

$$\mathbf{g} = \begin{pmatrix} (1 - \nabla^2) & 0 \\ 0 & -1 \end{pmatrix} \quad (6)$$

is the logical one. In order to obtain Eqs. (3) and (4) we also need to define the so-called Poisson structure,

$$\mathbf{J}_x = \begin{pmatrix} (1 - \nabla^2)^{-1} \{ (\cdot), \nabla^2 \Phi + x \}, & (1 - \nabla^2)^{-1} \{ (\cdot), \sqrt{r} x \} \\ \{ \sqrt{r} x, (\cdot) \}, & \{ (\cdot), \Phi \} \end{pmatrix},$$

where

$$\chi = \begin{pmatrix} \Phi \\ P \\ \sqrt{r} \end{pmatrix}.$$

Notice that \mathbf{J}_χ is antisymmetric with respect to the defined inner product (i.e., $\langle a | J_\chi(b) \rangle = -\langle b | J_\chi(a) \rangle$), and satisfies the Jacobi condition. Indeed the metric is chosen as in (6), so that the Poisson structure is antisymmetric. The Hamiltonian

$$H = \frac{1}{2} \langle \chi | \chi \rangle = \frac{1}{2} \int \left(\Phi^2 + \nabla \Phi^2 - \frac{P^2}{r} \right) d^2 \mathbf{x} \quad (7)$$

yields the equations of motion in symplectic form:

$$\partial_t \chi^i = J^{ij} \frac{\delta H}{\delta \chi_j}.$$

Notice that the Hamiltonian as defined in (7) is an integral of motion, which can be illustrated in a simpler way by multiplying Eq. (1) by Φ and Eq. (2) by $-P/r$ and adding, to obtain

$$\begin{aligned} \partial_t \left(\Phi^2 + \nabla \Phi^2 - \frac{P^2}{r} \right) + \nabla \cdot \left(\hat{\mathbf{y}} (\Phi^2 + 2\Phi P) - 2\Phi \partial_t \nabla \Phi \right. \\ \left. + \Phi^2 \hat{\mathbf{z}} \times \nabla \nabla^2 \Phi - \frac{P^2}{r} \hat{\mathbf{z}} \times \nabla \Phi \right) = 0. \end{aligned} \quad (8)$$

Here the first two terms are the usual potential and kinetic energy terms corresponding to the HM equation. The last term is an additional pressure contribution to the potential energy. The minus sign in front of this term is a manifestation of the fact that the model has unstable solutions (in that case, both Φ and P grow in such a way that the total Hamiltonian remains constant). The Talanov theorem⁵ predicts wave collapse in nonlinear Schrödinger-type equations when the total Hamiltonian is negative. For this model, the higher order nonlinear Schrödinger equation,⁶ that is derived with a reductive perturbation expansion, has the effective Hamiltonian (to second order in the expansion parameter ϵ)

$$\begin{aligned} H = \frac{1}{2} \int \left[|\partial_{\mathbf{k}} \cdot \nabla \bar{\Phi}(\mathbf{x})|^2 \omega(\mathbf{k}) + |\bar{\Phi}|^2 \right. \\ \left. - \left(\frac{|\bar{\Phi}|^2 + \frac{k_y}{\omega} |\bar{P}|^2}{[2\omega(1+k^2) - k_y]} \right) \omega(\mathbf{k}) \right] d^3 \mathbf{x}. \end{aligned}$$

Here $\partial_{\mathbf{k}}$ acts on $\omega(\mathbf{k})$ and $\bar{\Phi}$ and \bar{P} are related to $\Phi(\mathbf{x})$ through the mean field equations in which the average Reynolds stress generated by the fluctuations drive the mean fields [e.g., $\bar{\Phi} \propto \hat{\mathbf{z}} \times \mathbf{k} \cdot \nabla (|\Phi|^2)$]. The form of this Hamiltonian is the same as the second order approximation to the average of the exact Hamiltonian given above, although the coefficients are calculated using conventional perturbation methods instead of directly expanding the metric and the Poisson structure, as well as the Hamiltonian. The coefficients in front of the mean fields are positive for both branches. However, the dispersion effects [e.g., $\partial_{\mathbf{k}\mathbf{k}} \omega(\mathbf{k})$] are mainly negative for the dominant branch. The $|\bar{\Phi}|^2$ term that

comes from $\{\Phi, \nabla^2 \Phi\}$ nonlinearity acts to reverse the sign of the Hamiltonian from its linear value and thus is the source of wave collapse.

There is, however, another constant of motion, which may be of special interest as a check on numerical calculations. This can be derived by multiplying Eq. (1) by P and Eq. (2) by $\Phi - \nabla^2 \Phi + P/r$ and adding to obtain:

$$\begin{aligned} \partial_t \left(2P(1 - \nabla^2) \Phi + \frac{P^2}{r} \right) + \nabla \cdot \left[-r \hat{\mathbf{y}} \left(\Phi^2 + \nabla \Phi^2 - \frac{P^2}{r} \right) \right. \\ \left. + 2r \nabla \Phi \partial_y \Phi + \hat{\mathbf{z}} \times \nabla \Phi \left(2P\Phi - 2P\nabla^2 \Phi + \frac{P^2}{r} \right) \right] = 0. \end{aligned} \quad (9)$$

This is an additional conservation law, which implies that the integral,

$$\begin{aligned} I_2 = \int \left(2P(1 - \nabla^2) \Phi + \frac{P^2}{r} \right) dV \\ = \int \left(\left[\Phi - \nabla^2 \Phi + \frac{P}{r} \right]^2 - [\Phi - \nabla^2 \Phi]^2 \right) r dv \end{aligned}$$

over the whole domain, is an exact and independent conserved quantity of the ETG model if the flux terms vanish on the boundary. It may be speculated that this second integral of motion is related to the enstrophy generated by the pressure dynamics.

In this Brief Communication we have investigated the Hamiltonian structure of the simple slab model of toroidal ETG. We have derived the symplectic Hamiltonian equations of motion, and found the form of the Hamiltonian. These can be used to calculate the multiple wave coupling coefficients corresponding to the particular model, more easily. Although everything that can be done with the symplectic equations can also be done with standard reductive perturbation methods, it provides a compact reformulation of the problem which may be useful. Naturally, the total Hamiltonian that is found is an integral of motion of the model equations as well. The Hamiltonian can be negative (or have opposite sign from its linear value) if the initial mean fields are sufficiently high. This indicates the possibility of wave collapse, from the Talanov theorem point of view. We have also found another independent integral of motion, which can be used to check numerical calculations.

ACKNOWLEDGMENTS

We would like to thank S. Champeaux, T. S. Hahm, and W. Nevins for valuable discussions.

This research was supported by U.S. Department of Energy Grant No. FG03-88ER 53275.

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